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n-Widths of H^p-Spaces in $L_q(-1, 1)$†

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The Kolmogorov n-widths d_n, Gel'fand n-widths d^n, and linear n-widths δ_n of the Hardy spaces H^p in $L_q(-1, 1)$ are estimated. It is shown that for $1 \le q < p \le \infty$ there exist constants k_1, k_2 such that

$$k_1 n^{(1/2)(1/q-1/p)} \exp\left(-\pi \sqrt{\frac{n}{2}\left(\frac{1}{q} - \frac{1}{p}\right)}\right)$$
$$\le d_n(H^p, L_q(-1, 1)),\ d^n(H^p, L_q(-1, 1)),\ \delta_n(H^p, L_q(-1, 1))$$
$$\le k_2 n^{1/2q} \exp\left(-\pi \sqrt{\frac{n}{2}\left(\frac{1}{q} - \frac{1}{p}\right)}\right).$$

Furthermore the n-widths of $H_p^* = \{f$ analytic in the unit disk Δ: $h(z) = f(z)/(1 - z^2) \in H^p\}$, $\|f\|_p^* = \|h\|_{H^p}$ in $L_\infty(-1, 1)$ are determined asymptotically. Let $1 < p < \infty$ and $p' = p/(p - 1)$. Then

$$k_1 n^{-1/2p} \exp\left(-\pi \sqrt{\frac{n}{2p'}}\right)$$
$$\le d_n(H_p^*, L_\infty(-1, 1)),\ d^n(H_p^*, L_\infty(-1, 1)),\ \delta_n(H_p^*, L_\infty(-1, 1))$$
$$\le k_2 \exp\left(-\pi \sqrt{\frac{n}{2p'}}\right).$$

These results are improvements of estimates previously obtained by Burchard and Höllig and by Stenger.

1. Introduction and Summary

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be arbitrary Banach spaces and let us assume that X is continuously imbedded into Y by a continuous imbedding operator i: $X \to Y$.

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The Kolmogorov n-widths d_n of X in Y are defined by

$$d_n(X, Y) = \inf_{Y_n} \sup_{\|x\|_x \leq 1} \inf_{y \in Y_n} \|x - y\|_Y,$$

where Y_n runs over all subspaces of Y of dimension n or less.

The Gel'fand n-widths d^n of X in Y are defined by

$$d^n(X, Y) = \inf_{X_n} \sup_{\substack{\|x\|_x \leq 1 \\ x \in X_n}} \|x\|_Y,$$

where the infimum is taken over all subspaces X_n of codimension at most n.

The linear n-widths of X in Y are given by

$$\delta_n(X, Y) = \inf_{P_n} \sup_{\|x\|_x \leq 1} \|x - P_n x\|_Y,$$

where P_n is any continuous linear operator of X into Y of rank at most n.

As a consequence of the definitions we have always

$$d_n(X, Y),\ d^n(X, Y) \leq \delta_n(X, Y).$$

In the present paper we study the n-widths of certain Hardy space imbeddings. Recall that the Hardy space H^p is the class of analytic functions in the unit disk $\Delta \subset \mathbb{C}$ for which

$$\|f\|_{H^p} = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(r\, e^{i\theta})|^p \, d\theta\right)^{1/p}, \qquad 1 \leq p < \infty$$

$$\|f\|_{H^\infty} = \sup_{|z|<1} |f(z)|$$

is finite. It is well known that for an arbitrary compact subset E of Δ we have

$$\lim_{n\to\infty} (d_n(H^\infty, C(E))^{1/n} = \exp\left(-\frac{1}{\operatorname{cap}(E, \Delta)}\right).$$

Here $C(E)$ is the space of continuous functions on E and $\operatorname{cap}(E, \Delta)$ is the Green's capacity of E with respect to Δ (for details see Fisher and Micchelli, 1980, and the references given therein). In the case of an interval $[a, b] \subset (-1, 1)$ a more precise estimate is possible. There exist constants k_1, k_2 such that

$$k_1 \exp\left(-\frac{n}{\operatorname{cap}([a, b], \Delta)}\right) \leq d_n(H^\infty, C[a, b]) \leq k_2 \exp\left(-\frac{n}{\operatorname{cap}([a, b], \Delta)}\right).$$

If we let a tend to -1 and b to 1, then the imbedding $H^\infty \to C(-1, 1)$ remains continuous but is no longer compact and the n-widths are constant: $d_n(H^\infty, C(-1, 1)) = 1$ for all $n \in \mathbb{N}$. If we replace, however, $C(-1, 1)$ by $L_q(-1, 1)$ with $q < \infty$, then the imbedding $H^\infty \to L_q(-1, 1)$ is again compact and it is interesting to ask how fast the n-widths decay to zero in this case.

Burchard and Höllig (1985) proved that for $1 \leq q < p \leq \infty$ there exist constants k_1, k_2, α, β such that

$$\begin{aligned} &k_1 n^\alpha \exp\left(-\pi \sqrt{n\left(\frac{1}{q} - \frac{1}{p}\right)}\right) \\ &\quad \leq d_n(H^p, L_q(-1, 1)), d^n(H^p, L_q(-1, 1)), \delta_n(H^p, L_q(-1, 1)) \\ &\quad \leq k_2 n^\beta \exp\left(-\frac{\pi}{2} \sqrt{n\left(\frac{1}{q} - \frac{1}{p}\right)}\right). \end{aligned}$$

This estimate shows a gap by a factor $\frac{1}{2}$ in the exponent. The first main result of the present paper exactly determines the size of the exponent.

THEOREM 1. *For $1 \leq q < p \leq \infty$ there exist constants k_1, k_2 such that*

$$\begin{aligned} &k_1 n^{(1/2)(1/q - 1/p)} \exp\left(-\pi \sqrt{\frac{n}{2}\left(\frac{1}{q} - \frac{1}{p}\right)}\right) \\ &\quad \leq d_n(H^p, L_q(-1, 1)), d^n(H^p, L_q(-1, 1)), \delta_n(H^p, L_q(-1, 1)) \\ &\quad \leq k_2 n^{1/2q} \exp\left(-\pi \sqrt{\frac{n}{2}\left(\frac{1}{q} - \frac{1}{p}\right)}\right). \end{aligned}$$

This theorem shows that both the upper and the lower estimate of Burchard and Höllig (1985) were not sharp. Additionally in the case $p = \infty$ the behavior of the polynomial factor is determined exactly.

The second part of the paper deals with the space

$$H_p^* = \left\{ f \text{ analytic in } \Delta\colon h(z) = \frac{f(z)}{1 - z^2} \in H^p \right\}, \qquad \|f\|_p^* = \|h\|_{H^p}.$$

The space H_p^* and its practical applications were intensively studied by Stenger (1978, 1986). Our second main result estimates the n-widths of H_p^* in $L_\infty(-1, 1)$.

THEOREM 2. *Let $1 < p < \infty$ and $p' = p/(p-1)$. Then there exist constants k_1, k_2 such that*

$$
\begin{aligned}
&k_1 n^{-1/2p} \exp\left(-\pi\sqrt{\frac{n}{2p'}}\right)\\
&\quad\le d_n(H_p^*, L_\infty(-1, 1)),\ d^n(H_p^*, L_\infty(-1, 1)),\ \delta_n(H_p^*, L_\infty(-1, 1))\\
&\quad\le k_2 \exp\left(-\pi\sqrt{\frac{n}{2p'}}\right).
\end{aligned}
$$

Again this result is an improvement of a result of Burchard and Höllig (1985) who could only prove the weaker estimate

$$
\begin{aligned}
&k_1 n^{\alpha} \exp\left(-\pi\sqrt{\frac{n}{p'}}\right)\\
&\quad\le d_n(H_p^*, L_\infty(-1, 1)),\ d^n(H_p^*, L_\infty(-1, 1)),\ \delta_n(H_p^*, L_\infty(-1, 1))\\
&\quad\le k_2 n^{\beta} \exp\left(-\frac{\pi}{2}\sqrt{\frac{n}{p'}}\right).
\end{aligned}
$$

In the proof of our upper bound for $\delta_n(H_p^*, L_\infty(-1, 1))$ we construct a rational approximation of order n, which linearly depends on the function that is being approximated. This kind of approximation was studied in detail in Stenger (1986). The rational approximations of the present paper possess the same basic structure than those constructed in Stenger (1986). However, we obtain an exponential rate of decay to zero, which is faster by a factor $\sqrt{2}$ than in Stenger (1986).

The basic technique used by Burchard and Höllig (1985) consisted in mapping the unit disk conformally onto a horizontal strip and then approximating in the strip with weighted cardinal series. Our approach is quite different and is based on Blaschke products. We shall use the strong connection between n-widths and Blaschke products observed in Fisher and Micchelli (1980) as well as some results of Andersson (1980) obtained in the context of optimal quadrature.

In Section 2 we first deal with the case $p = \infty$ in Theorem 1. Then we prove in Sections 3 and 4 the general upper and lower bounds for Theorem 1, respectively. Finally, Section 5 contains the proof of Theorem 2. Since the Kolmogorov and Gel'fand n-widths never exceed the linear n-widths, it will be sufficient to prove upper estimates for the linear n-widths and lower estimates for the Kolmogorov and Gel'fand n-widths.

2. The H^∞-Case

The aim of this section is to prove Theorem 1 for the special case $p = \infty$. A Blaschke product of degree n is a function of the form

$$B(z) = \lambda \prod_{j=1}^{n} \frac{z - \alpha_j}{1 - \bar{\alpha}_j z}, \qquad |\alpha_j| < 1,\ j = 1, \ldots, n;\ |\lambda| = 1.$$

We let $\mathbb{B}_n$ denote the set of all Blaschke products of degree n or less.

Fisher and Micchelli (1980) proved the following interesting characterization of n-widths in terms of Blaschke products.

Lemma 1. *Let E be a compact subset of Δ and μ be a finite measure with support in E. Then we have for* $1 \le q < \infty$

$$d_n(H^\infty, L_q(\mu)) = d^n(H^\infty, L_q(\mu)) = \delta_n(H^\infty, L_q(\mu)) = \inf_{B \in \mathbb{B}_n} \|B\|_{L_q(\mu)}.$$

Using the compactness of $\mathbb{B}_n$ with respect to the topology of local uniform convergence on Δ one can extend Lemma 1 to the case $E = (-1, 1)$.

Corollary 2. *Lemma* 1 *holds for finite measures on* $(-1, 1)$. *In particular we have for standard Lebesgue measure*

$$d_n(H^\infty, L_q(-1, 1)) = d^n(H^\infty, L_q(-1, 1)) = \delta_n(H^\infty, L_q(-1, 1))$$
$$= \inf_{B \in \mathbb{B}_n} \|B\|_{L_q(-1,1)}.$$

The asymptotic behavior of the last infimum is known due to Andersson (1980).

Lemma 3. *There exist constants k_1, k_2 such that*

$$k_1 n^{1/2q} \exp\left(-\pi \sqrt{\frac{n}{2q}}\right) \le \inf_{B \in \mathbb{B}_n} \|B\|_{L_q(-1,1)}$$
$$\le k_2 n^{1/2q} \exp\left(-\pi \sqrt{\frac{n}{2q}}\right).$$

Corollary 2 and Lemma 3 yield the desired result of Theorem 1 for the special case $p = \infty$. Let us emphasize that Andersson was working in a quite different context when proving Lemma 3. He investigated optimal quadrature formulas in Hardy spaces. However, it turns out that his result can also be successfully applied to the determination of n-widths.

3. The General Upper Bound in Theorem 1

The starting point for the general upper estimate is again a result of Fisher and Micchelli (1980).

LEMMA 4. *For $1 \leq q < p \leq \infty$ we have*

$$\delta_n(H^p, L_q(-1, 1)) \leq \inf_{B \in \mathbb{B}_n} \left\| \frac{B(\zeta)}{(1 - \zeta^2)^{1/p}} \right\|_{L_q(-1,1)}.$$

Let us fix a parameter r in the interval $1/p < r < 1/q$ and insert a dummy factor $(1 - \zeta^2)^r$ in the relevant integral:

$$\begin{aligned} &\left\| \frac{B(\zeta)}{(1 - \zeta^2)^{1/p}} \right\|_{L_q(-1,1)} \\ &= \left(\int_{-1}^{1} \left| \frac{1}{(1 - \zeta^2)^r} (1 - \zeta^2)^{r-1/p} B(\zeta) \right|^q d\zeta \right)^{1/q} \\ &\leq \left(\int_{-1}^{1} \frac{1}{(1 - \zeta^2)^{r \cdot q}} d\zeta \right)^{1/q} \max_{\zeta \in [-1,1]} (1 - \zeta^2)^{r-1/p} |B(\zeta)|. \end{aligned}$$

It is elementary that the last integral is bounded by $(2/(1 - r \cdot q))^{1/q}$. In order to cope with the second factor we need a deep lemma of Ganelius (cf. Andersson, 1980).

LEMMA 5. *There exists $B \in \mathbb{B}_n$ such that*

$$\max_{\zeta \in [-1,1]} (1 - \zeta^2)^{r-1/p} |B(\zeta)| \leq C \exp\left(-\pi \sqrt{\frac{n}{2} \left(r - \frac{1}{p} \right)} \right).$$

Now choose the free parameter $r = (1/q)(1 - 1/\sqrt{n})$. Then we obtain

$$\delta_n(H^p, L_q(-1, 1)) \leq C n^{1/2q} \exp\left(-\pi \sqrt{\frac{n}{2} \left(\frac{1}{q} - \frac{1}{p} - \frac{1}{q\sqrt{n}} \right)} \right).$$

Finally Taylor expansion of $\sqrt{1/q - 1/p - 1/q\sqrt{n}}$ with respect to $1/q\sqrt{n}$ *yields the desired upper estimate in Theorem* 1.

4. The General Lower Bound in Theorem 1

The first step in the lower estimate consists in reducing the problem from H^p to H^∞. Let us introduce for this purpose the function $\Psi(\zeta) = 1/(1-\zeta^2)$. Then $\Psi^{1/p-\epsilon} \in H^p$ with

$$\|\Psi^{1/p-\varepsilon}\|_{H^p} \leq C\left(\int_0^{\pi/2} \sin^{\varepsilon\cdot p-1}(\theta)\, d\theta\right)^{1/p} \leq C\varepsilon^{-1/p}. \tag{4.1}$$

Furthermore $d\mu(x) = \Psi^{q/p-\varepsilon\cdot q}(x)\, dx$ is a finite measure on $(-1, 1)$ for $q < p$ and small ε.

We consider now the following composition of operators:

$$\begin{array}{ccccccc} H^\infty & \xrightarrow{M_1} & H^p & \xrightarrow{i} & L_q(-1, 1) & \xrightarrow{M_2} & L_q(\mu) \\ f \mapsto \Psi^{1/p-\varepsilon}\cdot f & & & & g \mapsto \Psi^{-1/p+\varepsilon}\cdot g. & & \end{array}$$

Here i is the imbedding operator, whose n-widths we want to estimate from below. For the norm of the multiplication operator M_1 we have $\|M_1\| \leq C\varepsilon^{-1/p}$ because of (4.1) and by the definition of μ the multiplication operator M_2 is an isometry.

The factorization property of the n-widths implies (cf. Pietsch, 1980) that

$$d_n(M_2 \circ i \circ M_1) \leq \|M_2\| d_n(i) \|M_1\|.$$

Hence we obtain

$$C\varepsilon^{1/p} d_n(H^\infty, L_q(\mu)) \leq d_n(H^p, L_q(-1, 1)). \tag{4.2}$$

According to Corollary 2 the n-widths of H^∞ in $L_q(\mu)$ are given by

$$\begin{aligned} d_n(H^\infty, L_q(\mu)) &= \inf_{B\in\mathbb{B}_n} \|B\|_{L_q(\mu)} \\ &= \inf_{B\in\mathbb{B}_n} \|B(x)\, \Psi^{1/p-\varepsilon}(x)\|_{L_q(-1,1)}. \end{aligned} \tag{4.3}$$

Exactly the same analysis applies to the Gel'fand n-widths d^n.

We now aim to prove a lower estimate for the infimum in (4.3) by generalizing some ideas from Andersson (1980).

For $0 < \rho < 1$ let

$$C_\rho = \left(\log \frac{1+\rho}{1-\rho}\right)^{-1}$$

and

$$w(x) = w_\rho(x) = C_\rho \Psi(x) = C_\rho \frac{1}{1-x^2}.$$

Set for notational convenience $1/\tilde{p} = 1/p - \varepsilon$ and let $B \in \mathbb{B}_n$ be arbitrary. Since $\int_{-\rho}^{\rho} w(x)\,dx = 1$ Jensen's inequality implies that

$$\begin{aligned}
&\|B(x)\Psi^{1/\tilde{p}}(x)\|_{L_q(-1,1)}^q \\
&\geq \int_{-\rho}^{\rho} \left| \frac{B(x)}{(1-x^2)^{1/\tilde{p}}} \right|^q w(x)^{-1} w(x)\,dx \\
&= \int_{-\rho}^{\rho} |B(x)|^q w(x)^{-1+q/\tilde{p}} C_\rho^{-q/\tilde{p}} w(x)\,dx \\
&\geq \exp \int_{-\rho}^{\rho} \left[q \log|B(x)| - \left(1 - \frac{q}{\tilde{p}}\right) \log w(x) - \frac{q}{\tilde{p}} \log C_\rho \right] w(x)\,dx.
\end{aligned}$$

Let us denote the integral in the exponent by J. Newman (1979) and Andersson (1980) showed that

$$\int_{-\rho}^{\rho} q \log|B(x)| w(x)\,dx \geq -C_\rho q \frac{n\pi^2}{4}$$

and

$$\int_{-\rho}^{\rho} \log\,(w(x)) w(x)\,dx \leq \log C_\rho + C_\rho[1 + \tfrac{1}{2}\log^2(1-\rho)].$$

Observing that $C_\rho \leq 1/(-\log(1-\rho)) \leq C$ for $\rho \geq \frac{1}{2}$ with a constant C independent of ρ we obtain

$$J \geq \frac{1}{\log(1-\rho)} q \frac{n\pi^2}{4} + \log(-\log(1-\rho)) + C + \frac{1}{2}\left(1 - \frac{q}{\tilde{p}}\right) \log(1-\rho).$$

Choose now the parameter ρ such that

$$\log(1-\rho) = -\sqrt{n}\,\frac{\pi}{\sqrt{2}} \frac{1}{\sqrt{1/q - 1/\tilde{p}}}.$$

This choice of ρ yields

$$J \geq -\pi q \sqrt{\frac{n}{2}}\left(\frac{1}{q} - \frac{1}{\tilde{p}}\right) + \log\sqrt{n} + C.$$

Recalling (4.2), (4.3), and the definition of $1/\tilde{p}$ we get

$$\begin{aligned} d_n(H^p, L_q(-1, 1)) &\geq C\varepsilon^{1/p} \inf_{B \in \mathbb{B}_n} \|B(x)\Psi^{1/p-\varepsilon}(x)\|_{L_q(-1,1)} \\ &\geq C\varepsilon^{1/p} \exp\left(\frac{1}{q} J\right) \\ &\geq C\varepsilon^{1/p} n^{1/2q} \exp\left(-\pi \sqrt{\frac{n}{2}}\left(\frac{1}{q} - \frac{1}{p} + \varepsilon\right)\right). \end{aligned}$$

Taylor expansion of $\sqrt{1/q - 1/p + \varepsilon}$ with respect to ε and choosing $\varepsilon = 1/\sqrt{n}$ yields the desired lower estimate in Theorem 1.

5. Proof of Theorem 2

Let us first mention that the imbedding $H_p^* \to L_\infty(-1, 1)$ is well defined. This follows from the fact that for $h \in H^p$ and $\zeta \in (-1, 1)$ we have $|h(\zeta)| \leq (1 - \zeta^2)^{-1/p}\|h\|_{H^p}$ and therefore $|f(\zeta)| \leq (1 - \zeta^2)^{1-1/p}\|f\|_{H_p^*}$ for $f \in H_p^*$ and $\zeta \in (-1, 1)$.

For the proof of Theorem 2 we need the following auxiliary result.

Lemma 6. *For $-1 < \zeta < 1$ and $1 < p' < \infty$ we have*

$$I_\zeta := \left(\int_{\partial\Delta} \frac{1}{|z - \zeta|^{p'}} |dz|\right)^{1/p'} \leq C(p')(1 - \zeta^2)^{1/p'-1}.$$

Proof. Let us assume without loss of generality that $-1 < \zeta \leq 0$ and consider the fractional linear transformation

$$\Phi: z \mapsto w(z) = \frac{1 + z}{1 - z}.$$

Φ maps the unit disk Δ conformally onto the right halfplane H with $x = \Phi(\zeta) \in (0, 1]$. Observing that the inverse transformation of Φ is given by $z(w) = (w - 1)/(w + 1)$ we get

$$I_\zeta^{p'} = \int_{\partial H} \left| \frac{(w+1)\cdot(x+1)}{2(w-x)} \right|^{p'} \frac{2}{|w+1|^2} |dw|$$

$$\leq \int_{\partial H} \left| \frac{w+1}{w-x} \right|^{p'} \frac{2}{|w+1|^2} |dw| \qquad \text{since } 0 < x \leq 1.$$

In Stenger (1986) theorem B1 it is shown that

$$\int_{\partial H} |(w+1)/(w-x)|^{p'} \frac{2}{|w+1|^2} |dw| \leq C(p')x^{1-p'}.$$

By definition $x = (1+\zeta)/(1-\zeta)$ and thus

$$x^{1-p'} = \left(\frac{1+\zeta}{1-\zeta}\right)^{1-p'} \leq 4^{p'-1}(1-\zeta^2)^{1-p'}.$$

This proves Lemma 6. ■

We now utilize Lemma 6 to obtain an upper estimate for $\delta_n(H_p^*, L_\infty(-1, 1))$. For this purpose we consider the Blaschke product $B(z) = \prod_{j=1}^n (z - z_j)/(1 - \bar{z}_j z)$ with

$$\max_{\zeta\in[-1,1]} (1-\zeta^2)^{1/p'} |B(\zeta)| \leq C \exp\left(-\pi\sqrt{\frac{n}{2p'}}\right). \tag{5.1}$$

The existence of B is guaranteed by Lemma 5.

Let us further set

$$\beta(z) = (1 - z^2)B(z).$$

The residue theorem yields, for fixed $\zeta \in (-1, 1)$ and $f \in H_p^*$,

$$\frac{1}{2\pi i}\int_{\partial\Delta} f(z)\frac{\beta(\zeta)}{\beta(z)}\frac{1}{z-\zeta}\,dz = f(\zeta) - \sum_{j=1}^n \frac{f(z_j)\beta(\zeta)}{(\zeta - z_j)\beta'(z_j)}. \tag{5.2}$$

Since $f \mapsto (P_n f)(\zeta) := \sum_{j=1}^n [f(z_j)\beta(\zeta)/(\zeta - z_j)\beta'(z_j)]$ defines an n-dimensional operator P_n from H_p^* into $L_\infty(-1, 1)$ we obtain

$$\delta_n(H_p^*, L_\infty(-1, 1)) \leq \sup_{\|f\|_{H_p^*}\leq 1} \sup_{-1\leq\zeta\leq 1} \left| f(\zeta) - \sum_{j=1}^n \frac{f(z_j)\beta(\zeta)}{(\zeta - z_j)\beta'(z_j)} \right|.$$

We now apply successively (5.2), Hölder's inequality, Lemma 6, and (5.1) to get

$$
\begin{aligned}
&\delta_n(H_p^*, L_\infty(-1, 1)) \\
&\le \sup_{\|f\|_{H_p^*}\le 1} \sup_{-1\le\zeta\le 1} \left| \frac{1}{2\pi i} \int_{\partial\Delta} \frac{f(z)}{1-z^2} \frac{(1-\zeta^2)B(\zeta)}{B(z)} \frac{1}{z-\zeta}\, dz \right| \\
&\le \sup_{-1\le\zeta\le 1} (1-\zeta^2)|B(\zeta)| \left(\frac{1}{2\pi} \int_{\partial\Delta} \frac{1}{|z-\zeta|^{p'}} \,|dz| \right)^{1/p'} \\
&\le \sup_{-1\le\zeta\le 1} (1-\zeta^2)|B(\zeta)| C(1-\zeta^2)^{1/p'-1} \\
&\le C \exp\left(-\pi \sqrt{\frac{n}{2p'}}\right).
\end{aligned}
$$

This is exactly the desired upper bound in Theorem 2. Furthermore we have solved an important open problem from Stenger (1986). A central result of Stenger (1986) states:

Let f be in the unit ball of H_p^*, let N be a positive integer, and define h, $\hat{z}_j$, and $\beta(z)$ by

$$h = \pi[p'/(2N)]^{1/2}$$

$$\hat{z}_j = \frac{e^{j\cdot h}-1}{e^{j\cdot h}+1} \qquad \beta(z) = (1-z^2)\prod_{j=-N}^{N} \frac{z-\hat{z}_j}{1-\hat{z}_j z}.$$

Then

$$\sup_{-1\le\zeta\le 1} \left| f(\zeta) - \sum_{j=-N}^{N} \frac{f(\hat{z}_j)\beta(\zeta)}{(\zeta-\hat{z}_j)\beta'(\hat{z}_j)} \right| \le CN^{1/2p'} \exp\left(-\pi\sqrt{\frac{N}{2p'}}\right).$$

Stenger poses the question whether the last estimate can be improved. This is indeed the case. Setting $n = 2N$ we see that our approximation operator possesses the same form than that one used by Stenger. However, by a different choice of the knots $z_1, \ldots, z_{2N}$, we achieve the better order of convergence $\exp(-\pi\sqrt{N/p'})$ in contrast to Stenger's order *$\exp(-\pi\sqrt{N/2p'})$*.

It remains to prove a lower bound for the n-widths of H_p^* in $L_\infty(-1, 1)$. The idea of proof is similar to that in Section 4. We restrict ourself to the Kolmogorov n-widths, but the proof for the Gel'fand n-widths is exactly the same.

Let us fix $\varepsilon > 0$, set $q = (1 - \varepsilon)/(1/p' + \varepsilon)$, and define $\Psi(\zeta) = 1/(1 - \zeta^2)$. We consider now the following composition of operators:

$$\begin{array}{ccccccc} H^\infty & \xrightarrow{M_1} & H_p^* & \xrightarrow{i} & L_\infty(-1, 1) & \xrightarrow{M_2} & L_q(-1, 1) \\ f & \longmapsto & \Psi^{-1/p'-\varepsilon} \cdot f & & g & \longmapsto & \Psi^{1/p'+\varepsilon} \cdot g. \end{array}$$

For the norm of the multiplication operator M_1 we have by (4.1) that $\|M_1\| \leq C\varepsilon^{-1/p}$. The norm of the multiplication operator M_2 is bounded by $\|M_2\| \leq C\varepsilon^{-1/q}$. Hence the factorization property of the n-widths implies

$$C\varepsilon^{1/p}\varepsilon^{1/q} d_n(H^\infty, L_q(-1, 1)) \leq d_n(H_p^*, L_\infty(-1, 1)).$$

Using the lower bound for $d_n(H^\infty, L_q(-1, 1))$ proven in Theorem 1 we obtain

$$C\varepsilon^{1/p}\varepsilon^{1/q} n^{1/2q} \exp\left(-\pi \sqrt{\frac{n}{2q}}\right) \leq d_n\big(H_p^*, L_\infty(-1, 1)\big).$$

Finally, Taylor expansion of $1/\sqrt{q(\varepsilon)} = \sqrt{(1/p' + \varepsilon)/(1 - \varepsilon)}$ with respect to ε and choosing $\varepsilon = 1/\sqrt{n}$ yields the desired lower estimate in Theorem 2.

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